Division of Differential operators , intertwine relations and Darboux Transformations

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Abstract

The problem of a differential operator left- and right division is solved in terms of generalized Bell polinomials for nonabelian differential unitary ring . The definition of the polinomials is made by means of recurrent relations. The expresions of classic Bell polinomils via generalized one is given. The conditions of an exact factorization possibility leads to the intertwine relation and results in some linearizablegeneralized Burgers equation. An alternative proof of the Matveev theorem is given and Darboux - Matveev transformations formula for coefficients follows from the intertwine relations and also expressed in the generalized Bell polinomials.

1 Introduction

Problems of solitonics and, more general, of nonlinear partial differential equation (NLPDE) integration is simplifying when one introduce a special factorization of a (differential) linear operator associated with the problem. The technique was actively used in theoretical physics [1] as well as in mathematics from early [2] to [3]. First comes from attempts to build explicit solutions of the one-dimensional Schrödinger equation and leads to so-called "supersymmetry" and the second relates to general problems of a differential field theory. The "physical line" was developed for partial differential equations (PDE) in [4] where the direct connection with the theory of Darboux transformations was noticed. Both approaches connect factors with some Riccati equation that is a feature of the general factorization problem [5] of importance from algorithmic point of view [6]. The problem of factorization was succesfully investigated algebraically for a commutative field in [7,8] and non-abelian one in [9] that allows to find appropriate algorithm for computations [10]. Applications to nonlinear ordinary differential equations (NODE) decomposition theory shows a way to NLPDE via classical reductions of S. Lie (point symmetries). A search of approaches to solution of integrable NLDPE or looking for somewhat sort of a language in which algebraic algorithms could be formulated in a compact form lead us to the theory of Bell polinomials. The polinomials appears naturally in Darboux techniques [11] and even a disarmed eye see a special polinomial combinations of derivatives in famous integrable equations. This way having in mind factorization of Schrödinger operators and Cole-Hopf linearization of the Burgers equation the Bell polinomials in fact appear in many contexts and generalized for nonabelian coefficients at [12]. The last result leads to a nonabelian Burgers hierarchy, directly linearized by the Cole-Hopf substitution analogue. From the side of integrable nonlinear differential equations form and their bilinear (Hirota) counterparts the introduction of "binary" Bell polinomials allows to express the algorithm of linearization of a given NLDPE in compact combinatoric language [13].

In the Section 2 of this paper we, after some notations, introduce left and right nonabelian polinomials by recurrence relations. The polimomials correspond to [12] but the form of their introduction is adjusted to our purposes and give some new useful relations. We also prove similar relationships in our context. In the Secton 3 we make next step on a way of the generalization of bell polinomials. We introduce some auxiliary DO and show how to make a linear differential operator (LDO) division by left and right factors of the first order LDO and formulate conditions in which a factorization is possible (give a zero remainder) for a nonabelian field. Next section (4) links a solution of the problem of a factorization with generalized (and linearized by Cole-Hopf substitution analog) Riccati equation. At the last section we give a compact formulation and proof of the Matveev theorem [15,16] in our notations deriving the potenyials' (LDO coefficients) transforms in term sof Bell polinomials.

2 Main notations and auxiliary results. Bell polinomials.

Let K be a differential ring of the zero characteristics with unit e (i.e. unitary ring) and an involution denoted by a superscript *. The differentiation is denoted as D. The differentiation and the involution are in accordance with operations in K, i.e.

1.$(a^*)^* = a, (a+b)^* = a^* + b^*; (ab)^* = b^* a^*, a, b \in K$

2.$D(a+b) = Da + Db; D(ab) = (Da)b + aDb;$

$$3.(Da)^* = -Da^*; \tag{1}$$

4. the D^n operators form a basis in K-module Diff(K) of differential operators. The subring of constants is K_0 and a multiplicative group of elements is G.

5. For any $s \in K$ there exists an element $\varphi \in K$ such that $D\varphi = s\varphi$, that means also the existence of a solution of the equation

$$D\phi = -\phi s \tag{2}$$

due to the involution properties.

There are lot of applications in the theory of integrable nonlinear equations and quantum problems connected with rings of square matrices. In this case the matrices are parametrized by a variable x and D may be a derivative with respect to this variable or a combination of partial derivatives that satisfy the conditions (3,4). If D is such usual differentiation, then the involution "*" is the Hermitian conjugation.In the case of a commutator $Da = [d, a]$, $(Da)^* = -[d^*, a]$. Having in mind such or similar applications we shall name the involution as conjugation. We do not restrict oneself by the matrix-valued case: an appropriate operator ring also good for our theory.

Below we introduce left and right nonabelian Bell poplinomials (see also [12] The left differential Bell polinomials are defined by Definiton 1

$$B_0(s) = e \tag{3}$$

and the recurrent relation

$$B_n(s) = DB_{n-1}(s) + B_{n-1}(s)s, n = 1, 2, .. \tag{4}$$

Statement 1. *If the element* $\varphi \in G$ *satisfy the equation* $D\varphi = s\varphi$ *then*

$$D^n \varphi = B_n(s)\,\varphi, \quad n = 0, 1, 2, \ldots . \tag{5}$$

Proof. Let us use the induction. At $n = 0, 1$ the equality (5) is trivial, from (3,4) $n = 1$ we get

$$B_1(s) = s;$$

therefore,

$$D\,\varphi = s\,\varphi = B_1(s)\,\varphi. \tag{6}$$

Here and further we shall denote $Dx = x', x \in K$ for brevity especially when the operator D acts only on the nearest element. Let the equality (5) be trivial for some n, then, by means of the relations (6), (8) we obtain

$$D^{n+1}\,\varphi = D(B_n(s)\,\varphi) = B_n'(s)\,\varphi + B_n(s)\,D\,\varphi = B_n'(s)\,\varphi + B_n(s)\,s\,\varphi =$$

$$(D\,B_n(s) + B_n(s)\,s)\,\varphi = B_{n+1}(s)\,\varphi;$$

it means that (5) take place for the next value of $n+1$, hence for all the natural n or zero. $\square$

Evaluations by (6) give

$$B_1(s) = s;\;\; B_2(s) = s^2 + Ds;\;\; B_3(s) = s^3 + 2s'\,s + s\,Ds + D^2 s;$$
$$B_4(s) = s^4 + 3s'\,s^2 + 2s\,s'\,s + s^2\,Ds + 3s''\,s + 3(Ds)^2 + s\,D^2 s + D^3 s.$$

Right Bell polinomials are introduced similarily.

Definition 2 *The right nonabelian differential Bell polinomials* are defined by the equalty

$$B_0^+(s) = e \tag{7}$$

and *a recurrence*

$$B_n^+(s) = -DB_{n-1}^+(s) + s\,B_{n-1}^+(s), \quad n = 1, 2, \ldots. \tag{8}$$

Statement 2. *If the element* $\phi \in G$ *satisfies the equation* (2), *then*

$$D^n\,\phi = (-1)^n\,\phi\,B_n^+(s), \quad n = 0, 1, 2, \ldots. \tag{9}$$

Proof. Treat the induction again. At $n = 0,\ 1$ the equality (8) trivializes and $n = 1$ yields (2), whence (7), (8) give

$$B_1^+(s) = s;$$

Otherwise,

$$D\,\phi = -\phi\,s = (-1)^1\,\phi\,B_1^+(s).$$

Let the equality (9) is valid for some n, then, by means of (2), (8), (10) we get

$$D^{n+1}\,\phi = D((-1)^n\,\phi\,B_n^+(s)) = (-1)^n\,(\phi'\,B_n^+(s) + \phi\,DB_n^+(s)) =$$
$$(-1)^n\,(-\phi\,s\,B_n^+(s) + \phi\,DB_n^+(s)) = (-1)^{n+1}\,\phi\,(-DB_n^+(s) + s\,B_n^+(s)) =$$
$$(-1)^{n+1}\,\phi\,B_{n+1}^+(s);$$

so, it is valid for the next $n+1$, hence for any natural or zero n. $\square$

Calculations by (8) give

$$B_1^+(s) = s;\;\; B_2^+(s) = s^2 - Ds;\;\; B_3^+(s) = s^3 - s'\,s - 2s\,Ds + D^2 s;$$
$$B_4^+(s) = s^4 - s'\,s^2 - 2s\,s'\,s - 3s^2\,Ds + s''\,s + 3(Ds)^2 + 3s\,D^2 s - D^3 s.$$

The right polinomials go to ones determined at [12] if change $D \to -D$. There exists a simple but useful link between the right and left Bell polinomials generated by the *conjugation*.

Statement 3. *The left and right Bell polinomials are connected by the following relations*:

$$B_n(s)^* = B_n^+(s^*), \quad B_n^+(s)^* = B_n(s^*). \tag{11}$$

Proof. It is enough to prove the first one as the second is conjugate. For this proof the induction again starts from a trivial point $n = 0$ of the equality (11), use the recurrences (4), (8) and give

$$B_{n+1}(s)^* = -DB_n(s)^* + s^*\,B_n(s)^* = -DB_n^+(s^*) + s^*\,B_n^+(s^*) = B_{n+1}^+(s^*).$$

That means the validity for any n. $\square$

If the ring is abelian, left and right polinomials coincide..

Remark 1. **The statement 4** means, that for the Bell polinomials takes place *a duality*: *any relation connecting right BP goes to one connecting the left BP and vice versa*. Let us denote further

$$L_s = D - s. \tag{12}$$

Note that the recursion (7) may be written by means of the designation (12) as

$$B_{n+1}^+(s) = -L_s\,B_n^+(s), \quad n = 0, 1, 2, \ldots;$$

with the simple corollary

$$B_n^+(s) = (-1)^n\,L_s^n\,e, \quad n = 0, 1, 2, \ldots. \tag{13}$$

3 One more generalization of Bell polinimials

In the next section the problem of division of an arbitrary operator L by the operator L_s will be studied. For the solution of it, we introduce auxiliary operators for a right division H_n by means of

Definition 3

$$H_0 = e. \tag{14}$$

and *the recurrence::*

$$H_n = D\, H_{n-1} + B_n(s), \quad n = 1,\, 2,\, \dots . \tag{15}$$

Statement 5. *The identity holds*

$$D^n = H_{n-1}\, L_s + B_n(s), \quad n = 1, 2, \dots . \tag{16}$$

Proof. The induction gives the following. At $n = 1$ the equality (16) is trivial. Let it be for n, then

$$D^{n+1} = D\,(H_{n-1}\, L_s{+}B_n(s)) = D\, H_{n-1}\, L_s{+}D\, B_n(s) = D\, H_{n-1}\, L_s{+}B_n(s)\, D{+}DB_n(s) =$$

$$D\, H_{n-1}\, L_s + B_n(s)\, L_s + DB_n(s) + B_n(s)\, s = (D\, H_{n-1} + B_n(s))\, L_s + B_{n+1}(s) =$$

$$H_n\, L_s + B_{n+1}(s),$$

it is (16), but after the change n for $n+1$. Hence, the equality (16) is valid for all natural n. □

Coefficients of the operators H_n are expressed via *generalized Bell polinomials* , that are defined by

bf Definition 4

$$B_{n,\,0}(s) = e, \quad n = 0, 1, 2, \dots , \tag{17}$$

and recurrence relations

$$B_{n,\,k}(s) = B_{n-1,\,k}(s) + DB_{n-1,\,k-1}(s), \quad k = \overline{1,\, n-1}, \quad n = 2, 3, \dots . \tag{18}$$

$$B_{n,\,n}(s) = DB_{n-1,\,n-1}(s) + B_n(s), \quad n = 1, 2, \dots . \tag{19}$$

Statement 6. *Generalized Bell polinomials are coefficients of the operators* H_n*, i.e.*

$$H_n = \sum_{k=0}^{n} B_{n,\,n-k}(s)\, D^k, \quad n = 0, 1, 2, \dots . \tag{20}$$

Proof. When $n = 0$ the equality (22) trivializes $H_0 = e$. so that it is enough to establish for operators (20) the recurrence (15), by the following transformations:

$$H_n - D\, H_{n-1} - B_n(s) = \sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k - \sum_{k=0}^{n-1} D\, B_{n-1,\, n-1-k}(s)\, D^k - B_n(s) =$$

$$\sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k - \sum_{k=0}^{n-1} (B_{n-1,\, n-1-k}(s)\, D^{k+1} + DB_{n-1,\, n-1-k}(s)\, D^k) - B_n(s) =$$

$$\sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k - \sum_{k=0}^{n-1} B_{n-1,\, n-1-k}(s)\, D^{k+1} - \sum_{k=0}^{n-1} DB_{n-1,\, n-1-k}(s)\, D^k - B_n(s) =$$

$$\sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k - \sum_{k=1}^{n} B_{n-1,\, n-k}(s)\, D^k - \sum_{k=0}^{n-1} DB_{n-1,\, n-1-k}(s)\, D^k - B_n(s) =$$

$$(B_{n,\, 0}(s) - B_{n-1,\, 0})\, D^n + \sum_{k=1}^{n-1} (B_{n,\, n-k}(s) - B_{n-1,\, n-k}(s) - DB_{n-1,\, n-1-k}(s))\, D^k +$$

$$(B_{n,\, n}(s) - DB_{n-1,\, n-1}(s) - B_n(s)) = (e - e)\, D^n = 0.$$

As the equality (14) and the recurrence (15) define the operators H_n, in unique way, the result of the calculations means the validity of (20). $\square$

The formulas (18) and (19) are simple, but nor very convenient for evaluations of $B_{n,\, k}(s)$
(you forced to go "ladder" way),therefore we suggest more complicated but practically easier algorithm. For this we put the representation (20) into (18); getting

$$0 = D^{n+1} - H_n\, L_s - B_{n+1}(s) = D^{n+1} - \sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k\, (D - s) - B_{n+1}(s) =$$

$$D^{n+1} - \sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^{k+1} + \sum_{k=0}^{n} B_{n,\, n-k}(s)\, D^k\, s - B_{n+1}(s) =$$

$$D^{n+1} - \sum_{k=1}^{n+1} B_{n,\, n-k+1}(s)\, D^k + \sum_{k=0}^{n} B_{n,\, n-k}(s) \sum_{i=0}^{k} \binom{k}{i} D^{k-i} s\, D^i - B_{n+1}(s) =$$

$$(e - B_{n,\, 0}(s))\, D^{n+1} - \sum_{k=1}^{n} B_{n,\, n-k+1}(s)\, D^k + \sum_{0 \le i \le k \le n} \binom{k}{i} B_{n,\, n-k}(s)\, D^{k-i} s\, D^i - B_{n+1}(s) =$$

$$(e - B_{n,\, 0}(s))\, D^{n+1} - \sum_{k=1}^{n} B_{n,\, n-k+1}(s)\, D^k + \sum_{0 \le k \le i \le n} \binom{i}{k} B_{n,\, n-i}(s)\, D^{i-k} s\, D^k - B_{n+1}(s) =$$

$$(e - e)\, D^{n+1} - \sum_{k=1}^{n} B_{n,\, n-k+1}(s)\, D^k + \sum_{k=0}^{n} \sum_{i=k}^{n} \binom{i}{k} B_{n,\, n-i}(s)\, D^{i-k} s\, D^k - B_{n+1}(s) =$$

$$\sum_{k=1}^{n} \left(-B_{n,\, n-k+1}(s) + \sum_{i=k}^{n} \binom{i}{k} B_{n,\, n-i}(s)\, D^{i-k} s \right) D^k + \left(\sum_{i=0}^{n} B_{n,\, n-i}(s)\, D^i s - B_{n+1}(s) \right).$$

The following formulas are extracted:

$$B_{n,\,n-k+1}(s) = \sum_{i=k}^{n} \binom{i}{k} B_{n,\,n-i}(s)\, D^{i-k} s, \quad k = \overline{1,\, n}, \;\; n = 0, 1, 2, \dots ; \tag{21}$$

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,\,n-i}(s)\, D^{i} s, \quad n = 0, 1, 2, \dots . \tag{22}$$

The formulae (22) expresses the standard (nonabelian) Bell polinomials via the generalized ones; it may be rewrited as

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,\,i}(s)\, D^{n-i} s, \quad n = 0, 1, 2, \dots . \tag{22}$$

If in the formulae (22) rearrange the summation by $k \to n - k + 1$, then after simple work yields:

$$B_{n,\,k}(s) = \sum_{i=0}^{k-1} \binom{n-i}{n-k+1} B_{n,\,i}(s)\, D^{k-i-1} s, \quad k = \overline{1,\, n}, \;\; n = 0, 1, 2, \dots . \tag{23}$$

Evaluation of the generalized Bell polinomials by (22) gives

$$B_{n,\,1}(s) = s; \quad B_{n,\,2}(s) = s^2 + n\, Ds; \quad B_{n,\,3}(s) = s^3 + n\, s'\, s + (n-1)\, s\, Ds + \binom{n}{2} D^2 s;$$

$$B_{n,\,4}(s) = s^4 + n\, s'\, s^2 + (n-1)\, s\, s'\, s + (n-2)\, s^2\, Ds + \binom{n}{2} s''\, s + n(n-2)\, (Ds)^2 +$$

$$\binom{n-1}{2} s\, D^2 s + \binom{n}{3} D^3 s.$$

For a solution of problem of the left division of a differential operator, L by L_s the similar but simpler consideration is necessary. The analog of the **Statement 5** is

Statement 7. *The following identity is valid*

$$D^n = L_s\, H^+_{n-1} + B^+_n(s), \quad n = 1, 2, \dots , \tag{27}$$

where

$$H^+_n = \sum_{k=0}^{n} B^+_{n-k}(s)\, D^k, \quad n = 0, 1, 2, \dots . \tag{28}$$

Proof. Let us solve the problem of the left division of the operator D^n by the operator L_s i.e. put

$$D^n = L_s\, H^+_{n-1} + r, \tag{29}$$

where r –remainder, and

$$H^+_{n-1} = \sum_{k=0}^{n-1} b_k\, D^k. \tag{30}$$

The remainder, r and the coefficients b_k are to be determined.

Substituting (30) into (29) and, transforming, we arrive at

$$L_s\, H^-_{n-1} + r = (D - s) \sum_{k=0}^{n-1} b_k\, D^k + r = \sum_{k=0}^{n-1} D\, b_k\, D^k - \sum_{k=0}^{n-1} s\, b_k\, D^k + r =$$

$$\sum_{k=0}^{n-1} (b_k\, D^{k+1} + Db_k\, D^k) - \sum_{k=0}^{n-1} s\, b_k\, D^k + r = \sum_{k=0}^{n-1} b_k\, D^{k+1} + \sum_{k=0}^{n-1} Db_k\, D^k - \sum_{k=0}^{n-1} s\, b_k\, D^k + r =$$

$$\sum_{k=1}^{n} b_{k-1}\, D^k + \sum_{k=0}^{n-1} (Db_k - s\, b_k)\, D^k + r = (Db_0 - s\, b_0 + r) + \sum_{k=1}^{n-1} (b_{k-1} + Db_k - s\, b_k)\, D^k + b_{n-1}\, D^n =$$

$$(L_s\, b_0 + r) + \sum_{k=1}^{n-1} (b_{k-1} + L_s\, b_k)\, D^k + b_{n-1}\, D^n.$$

Comparing the result with the left-hand side of (31),we obtain

$$b_{n-1} = e; \quad b_{k-1} = -L_s\, b_k, \quad k = \overline{1,\, n-1}; \quad r = -L_s\, b_0.$$

Recalling (7) and the recurrence (8), we represents:

$$b_k = B^+_{n-k-1}(s), \quad k = \overline{0,\, n-1}; \quad r = B^+_n(s).$$

The equalities (24) and (25) are their obvious corollaries. □

3. Division and factorization of differential operators. Generalized Riccati equations.

Let

$$L = \sum_{n=0}^{N} a_n\, D^n, \quad a_n \in K, \; - \tag{28}$$

be a differential operator of the order N. We shall study a right and left division of L by the operator L_s,, defined by the equality (12), Suppose

$$L = M\, L_s + r, \tag{29}$$

$$L = L_s\, M^+ + r^+, \tag{30}$$

where M, M^+ –results of right and left division and r, r^+ – remainders.

The statement 5 allows to solve the problem in a simple way.

Statement 8. *If the representation* (29), *is valid, then for the remainder* r *and the result* M *yields*:

$$r = \sum_{n=0}^{N} a_n\, B_n(s); \tag{31}$$

$$M = \sum_{n=1}^{N} a_n\, H_{n-1}, \tag{32}$$

or

$$M = \sum_{n=0}^{N-1} b_n\, D^n, \tag{33}$$

where

$$b_n = \sum_{k=n+1}^{N} a_k\, B_{k-1,\,n}(s), \quad n = \overline{0,\, N-1}. \tag{34}$$

Proof. Multiplying both sides of the equality (16) by a_n and summing by n, one get (29), (31) – (33). After the substitution of the representation (22) into (34) it results

$$M = \sum_{n=1}^{N} a_n\, H_{n-1} = \sum_{n=1}^{N} a_n \sum_{k=0}^{n-1} B_{n-1,\,k}(s)\, D^k = \sum_{1 \le n \le N,\, 0 \le k \le n-1} a_n\, B_{n-1,\,k}(s)\, D^k =$$
$$\sum_{1 \le k \le N,\, 0 \le n \le k-1} a_k\, B_{k-1,\,n}(s)\, D^n = \sum_{n=0}^{N-1} \sum_{k=n+1}^{N} a_k\, B_{k-1,\,n}(s)\, D^n,$$

from what the formulas (33), (34) follow.

As a corollary one get

Statement 9. *For the linear operator* L *to be a right divisible by* L_s *without remainder* it is *necessary and enough, that* s *to be a solution of the differential equation*

$$\sum_{n=0}^{N} a_n\, B_n(s) = 0. \tag{35}$$

If this condition holds, the operator L *factors as* :

$$L = M\, L_s,$$

where the value of the operator M *is given by* (32) *or by the expressions* (33), (34).

The equation (35) is nonlinear. At $N = 2$ it is Riccati equation, therefore it is natural to name it as a *generalized right Riccati equation.* The right Riccati equation is generalized by means of the Statement 2.

Statement 10. *Let an invertible function* φ *be a solution to the linear differential equation*

$$\sum_{n=0}^{N} a_n D^n \varphi = 0. \tag{36}$$

Then the operator L, *defined by the equality* (31), *is right divisible by* L_s, *where*

$$s = \varphi' \varphi^{-1}.$$

Morover s *is a solution of the right Riccati equation* (35).

For the solution of a left-division problem letus write a result in a form

$$M^+ = \sum_{n=0}^{N-1} b_n^+ D^n, \tag{37}$$

looking for the determination of b_n^+, $n = \overline{0,\, n-1}$.

Substitute the representation (37) into the right-hand side of the equation (30). By means the calculus analogues to that was used for the proof of the Statement 7, you obtain:

$$b_{N-1}^+ = a_N; \tag{38}$$

$$b_n^+ = a_{n+1} - L_s\, b_{n+1}^+, \quad n = \overline{0,\, N-2}; \tag{39}$$

$$r^+ = a_0 - L_s\, b_0^-. \tag{40}$$

Solving subsequently the equations (37) – (40) one goes to

$$b_n^+ = \sum_{k=n+1}^{N} (-1)^{k-n-1} L_s^{k-n-1} a_k, \quad n = \overline{0,\, N-1}; \tag{41}$$

$$r^+ = \sum_{k=0}^{N} (-1)^k L_s^k a_k. \tag{42}$$

The entities b_n^+, $n = \overline{0,\, N-1}$; r^+ may be expressed in terms of thr right Bell polinomials if use the equality (12) and take into account

$$L_s^k a = L_s^k e\, a = (-1)^k B_k^+(s)\, a.$$

Hence links (41), (42) transform to:

$$b_n^+ = \sum_{k=n+1}^{N} B_{k-n-1}^+(s)\, a_k, \quad n = \overline{0,\, N-1}; \tag{43}$$

$$r^+ = \sum_{k=0}^{N} B_k^+(s)\, a_k. \tag{44}$$

Formulas (30), (37), (43), (44) (or (41), (42)) give a solution of the left division problem of L by L_s . So , it is proved

Statement 11. *if the representation* (31), *is valid, then for the reminder* r^+ *and the result* M^+ *the formulas* (44) (*or* (42)) *and* (37), *take place and for the coefficients of the operator* M^+ *there is a representation* (43) (*or* (41)).

The straight corollary of this sentence is the following

Statement 12. *For the operator* L *to be left divisible by the operator* L_s (without remainder) *it is necessary and enough, that* s *be a solution of the differential equation*

$$\sum_{k=0}^{N} B_k(s)^+ \, a_k = 0. \tag{45}$$

if this condition holds, then the operator L *factorizes as*:

$$L = L_s \, M^+,$$

where the value of the operator M^+ *is given by the expressions* (37), (43) (*or* (42)).

The nonlinear equation (45) is called as *generalized left Riccati equation*.
The left Riccati equation is linearized obviously by the Statement 3. As a result we have

Statement 13. *Let an invertible function* φ *satisfy to the linear differential equation*

$$\sum_{n=0}^{N} (-1)^n \, B_n(s)^+ \, a_n \, D^n \varphi = 0. \tag{46}$$

Then the operator L, *determined by the equality* (31), *is left divisible by the operator* L_s, *where*

$$s = -\varphi^{-1} \, \varphi'.$$

The function s *is a solution to the left Riccati equation* (45).

Remark 2. Following F. Calogero classification, the Riccati equations (38), (45) are *C–integrable dynamical systems.*

4. Darboux transformation. Generalized Burgers equations.

We shall show that the problem under consideration of the operator division is connected directly with *Darboux transformation*. For the purpose we take a version in which at the ring K there exists one more differentiation D_0, which *commute* with the operator D D,

$$D_0 \, D = D \, D_0.$$

It may be a differentiation by a parameter t.

Let us list auxiliary commutation relation.

$$L_s \, r = r \, L_s + Dr + [r, \, s]. \tag{47}$$

Really,

$$L_s\, r - r\, L_s = (D-s)\, r - r\,(D-s) = D\, r - sr - r\, D + rs = r\, D + Dr - sr - r\, D + rs = Dr + [r,\, s].$$

Taking into account the equalities (48) and (31), we calculate

$$L_s\,(D_0 - L) = L_s\, D_0 - L_s\, L = D_0\, L_s + D_0 s - L_s\,(M\, L_s + r) = D_0\, L_s + D_0 s - L_s\, M\, L_s - L_s\, r =$$

$$D_0\, L_s + D_0 s - L_s\, M\, L_s - r\, L_s - Dr - [r,\, s] = (D_0 - L_s\, M - r)\, L_s + D_0 s - Dr - [r,\, s].$$

and arrive to the following relation:

$$L_s\,(D_0 - L) = (D_0 - \tilde{L})\, L_s + D_0 s - Dr - [r,\, s], \tag{48}$$

where

$$\tilde{L} = L_s\, M + r. \tag{49}$$

We results in the important conclusion:

Statement 14. *If a function s satisfy the equation*

$$D_0 s = Dr + [r,\, s], \tag{50}$$

the operator L_s intertwine the operators $D_0 - L$ з $D_0 - \tilde{L}$,

$$L_s\,(D_0 - L) = (D_0 - \tilde{L})\, L_s. \tag{54}$$

Now we would obtain the explicit expression for $\tilde{L}$, in the terms of (49), (31), (32); namely

$$\tilde{L} = L_s\, M + r = (D - s)\sum_{n=1}^{N} a_n\, H_{n-1} + \sum_{n=0}^{N} a_n\, B_n(s) =$$

$$\sum_{n=1}^{N} D\, a_n\, H_{n-1} - \sum_{n=1}^{N} s\, a_n\, H_{n-1} + \sum_{n=0}^{N} a_n\, B_n(s) =$$

$$\sum_{n=1}^{N} (Da_n\, H_{n-1} + a_n\, D\, H_{n-1} - s\, a_n\, H_{n-1} + a_n\, B_n(s)) + a_0\, B_0(s) =$$

$$\sum_{n=1}^{N} (Da_n\, H_{n-1} + a_n\,(H_n - B_n(s) + B_n(s)) - s\, a_n\, H_{n-1}) + a_0\, e =$$

$$\sum_{n=1}^{N} (a_n'\, H_{n-1} + a_n\, H_n - s\, a_n\, H_{n-1}) + a_0.$$

Finally, the transfomed operator is

$$\tilde{L} = \sum_{n=1}^{N} (a'_n\, H_{n-1} + a_n\, H_n - s\, a_n\, H_{n-1})) + a_0. \tag{52}$$

Let us write the equation (50) in the explicit form by virtue of the formula (31).

$$Dr + [r,\, s] = D \sum_{n=0}^{N} a_n\, B_n(s) + \left[\sum_{n=0}^{N} a_n\, B_n(s),\, s\right] =$$

$$\sum_{n=0}^{N} D\, a_n\, B_n(s) + \sum_{n=0}^{N} [a_n\, B_n(s),\, s] =$$

$$\sum_{n=0}^{N} (a'_n\, B_n(s) + a_n\, DB_n(s)) + \sum_{n=0}^{N} (a_n\, B_n(s)\, s - s\, a_n\, B_n(s)) =$$

$$\sum_{n=0}^{N} (a'_n\, B_n(s) + a_n\, DB_n(s) + a_n\, B_n(s)\, s - s\, a_n\, B_n(s)) =$$

$$\sum_{n=0}^{N} (a'_n\, B_n(s) + a_n\, (DB_n(s) + B_n(s)\, s) - s\, a_n\, B_n(s)) =$$

$$\sum_{n=0}^{N} (a'_n\, B_n(s) + a_n\, B_{n+1}(s) - s\, a_n\, B_n(s)).$$

It is established that for the intertwine relation (51) validity it is necessary and enough that s should be a solution of the equation

$$D_0 s = \sum_{n=0}^{N} (Da_n\, B_n(s) + a_n\, B_{n+1}(s) - s\, a_n\, B_n(s)). \tag{53}$$

Remark 3. The equation (53) is nonlinear, but linearizable, i.e. (by Calogero classification) it is *C–integrable.* This equation (in different forms) was introduced at [12,16]. The form we suggest is most compact nad convenient for a further investigations, e.g. in the franework of bilineraization technique of [13].

Example.

Let

$$L = D^2.$$

Then

$$\tilde{L} = H_2 - s\, H_1 = B_{2,\,0}(s)\, D^2 + B_{2,\,1}(s)\, D + B_{2,\,2}(s) - s\, (B_{1,\,0}(s)\, D + B_{1,\,1}(s)) =$$

$$D^2 + s\, D + (s^2 + 2Ds) - s\, (D + s) = D^2 + 2Ds.$$

The equation (53) takes the form:

$$D_0 s = D^2 s + 2Ds\, s.$$

In the case of scalar functions it is known *Burgers equation*. By this reason and due to the eqaution are integrable by the Cole-Hopf transformation, the equation (53) is naturally named *generalized Burgers equation*.

Statement 15. *Let an invertible function* φ, *is a solution to linear differential equation*

$$D_0 \varphi = L\,\varphi. \tag{54}$$

Then the function s *satisfy the generalized Burgers equation* (56).

Proof. Let us note that

$$(D\,r + [r,\,s])\,\varphi = (L_s)\,r\,\varphi + r\,s\,\varphi = (L_s)\,r\,\varphi + r\,D\,\varphi = L_s\,(r\,\varphi) = L_s\,L\,\varphi.$$

Further, acting to the equation (53) by the operator L_s and accounting for the relations (45) and (2), (16), we have

$$0 = L_s\,(D_0\varphi - L\,\varphi) = L_s\,D_0\varphi - L_s\,L\,\varphi = (D_0\,L_s + D_0 s)\,\varphi - L_s\,L\,\varphi =$$

$$D_0\,L_s\,\varphi + D_0 s\,\varphi - (D\,r + [r,\,s])\,\varphi = (D_0 s - D\,r - [r,\,s])\,\varphi,$$

due to the existence of the inverse element for φ one obtain (50), that is equivalent to (52). □

The obvious corollary of the intertwine relation (51) and the statement16 is

Theorem (Matveev). *Let functions* ψ and φ *are solutions of the equations*

$$D_0\,\psi = L\,\psi, \quad D_0\,\varphi = L\,\varphi,$$

for an invertible φ *Then the function*

$$\tilde{\psi} = L_s\,\psi = D\,\psi - s\,\psi, \quad s = D\,\varphi\,\varphi^{-1}, \tag{55}$$

is a solution of the equation

$$D_0\,\tilde{\psi} = \tilde{L}\,\tilde{\psi}. \tag{56}$$

The last statement accomplishes the proof of Matveev theorem [15,16]. The equality (55) gives a representation of the transformed operator in terms of the generalized Bell polinomials. The explicit expression for the transformed coefficient is

$$a_N[1] = a_N,$$

$$a_k[1] = a_k + \sum_{n=k}^{N} [a_n B_{n,n-k} + (a_n' - s a_n) B_{n-1,n-1-k}], \;\; k = 0, ..., N-1.$$

5. Conclusion .

It is shown that the division procedure for linear differential operators naturaly leads to the solution of its factorisation problem thats links to intertwine relations and Darboux transformations. The representation constructed here may, perhaps, open new possibilities to build up and study new integrable systems. One of us

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Appendix. Evaluation of generalized Bell polinomials..

Calculations by the relation (26) give the following:

At $n = 1$

$$B_{n,1}(s) = \sum_{i=0}^{0} \binom{n-i}{-i} B_{n,i}(s)\,(D^{-i}s) = \binom{n}{0} B_{n,0}(s)\,(D^0 s) = s;$$

At $n = 2$

$$B_{n,2}(s) = \sum_{i=0}^{1} \binom{n-i}{1-i} B_{n,i}(s)\,(D^{1-i}s) = \binom{n}{1} B_{n,0}(s)\,(D^1 s) + \binom{n-1}{0} B_{n,1}(s)\,(D^0 s) =$$

$$n\,Ds + s\,s = s^2 + n\,Ds.$$

At $n = 3$

$$B_{n,3}(s) = \sum_{i=0}^{2} \binom{n-i}{2-i} B_{n,i}(s)\,(D^{2-i}s) =$$

$$\binom{n}{2} B_{n,0}(s)\,D^2 s + \binom{n-1}{1} B_{n,1}(s)\,D^1 s + \binom{n-2}{0} B_{n,2}(s)\,D^0 s =$$

$$\binom{n}{2} D^2 s + (n-1)\,s\,Ds + (s^2 + n\,Ds)\,s = s^3 + n\,Ds\,s + (n-1)\,s\,Ds + \binom{n}{2} D^2 s.$$

At $n = 4$

$$B_{n,4}(s) = \sum_{i=0}^{3} \binom{n-i}{3-i} B_{n,i}(s)\,(D^{3-i}s) =$$

$$\binom{n}{3} B_{n,0}(s)\,D^3 s + \binom{n-1}{2} B_{n,1}(s)\,D^2 s + \binom{n-2}{1} B_{n,2}(s)\,D^1 s +$$

$$\binom{n-3}{0} B_{n,3}(s)\,D^0 s =$$

$$\binom{n}{3} D^3 s + \binom{n-1}{2} s\,D^2 s + (n-2)\,(s^2 + n\,Ds)\,Ds +$$

$$\left(s^3 + n\,Ds\,s + (n-1)\,s\,Ds + \binom{n}{2} D^2 s\right) s =$$

$$\binom{n}{3} D^3 s + \binom{n-1}{2} s\,D^2 s + (n-2)\,s^2\,Ds + n(n-2)\,(Ds)^2 + s^4 + n\,Ds\,s^2 +$$

$$(n-1)\, s\, Ds\, s + \binom{n}{2} D^2 s\, s =$$

$$s^4 + n\, Ds\, s^2 + (n-1)\, s\, Ds\, s + (n-2)\, s^2\, Ds + \binom{n}{2} D^2 s\, s + n(n-2)\, (Ds)^2 + \binom{n-1}{2} s\, D^2 s +$$

$$\binom{n}{3} D^3 s.$$

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